Does the photon-diffusion coefficient depend on absorption?

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We investigate the controversy over the precise form of the photon diffusion coefficient and suggest that it is largely independent of absorption, i.e., $D_0 = v/3\mu_s'$. After presentation of the general theoretical arguments underlying this assertion, Monte Carlo simulations are performed and explicitly reveal that the absorption-independent diffusion coefficient gives better agreement with theory than the traditionally accepted photon diffusion coefficient, $D_{\mu_a} = v/3(\mu_s' + \mu_a)$. The importance of resolving this controversy for the proper characterization of the material optical properties is discussed. © 1997 Optical Society of America [S0740-3232(97)01912-1]

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1. INTRODUCTION

Human tissues are turbid media wherein the transport of near-infrared photons is modeled quite well by the diffusion equation. Recently, there has been substantial interest in using near-infrared light to probe human tissues for localized heterogeneities such as breast and brain tumors, and several algorithms based on the photon diffusion equation for detecting and characterizing tumors have been developed.[1–11]

Nevertheless, there has been some confusion in the community about the exact form of the light diffusion constant in media with both absorption and scattering. Specifically, in most derivations of the diffusion equation from the transport equation, the diffusion coefficient D is found to be $D_{\mu_a} = v/3(\mu_s' + \mu_a)$,[4,12–17] where μ_s' is the medium reduced scattering coefficient, μ_a is its absorption coefficient, and v is the speed of light in the medium. Recently, however, it has been suggested that D should be independent of absorption, i.e., $D_0 = v/3\mu_s'$.[18,19] Resolution of this problem is important for accurate determination of optical properties, particularly in media wherein absorption is large.

In this paper we use Monte Carlo techniques to suggest that the photon diffusion coefficient is independent of absorption. A theoretical discussion similar to that given by Furutsu and Yamada[18] is first developed to illustrate the differences in a particular transformation property of the photon transport equation and the photon diffusion equation. In this case it is argued that the transformation property should be preserved, and this is possible only when $D = D_0$. This analysis provides some reasons to reject the traditional photon diffusion coefficient. We then develop a Monte Carlo program to simulate the propagation of photons in infinite, homogeneous, and isotropic media with optical properties resembling those of human tissues. Comparing the Monte Carlo results and photon diffusion theory by using the two different diffusion coefficients, we quantitatively demonstrate that agreement between experiment and theory is improved when $D = D_0$.

Characterization of tissue optical properties is an important goal of diffuse optical tomography.[6,20–22] In some cases the agreement between diffusion theory and observation will depend on the exact nature of the diffusion coefficient. For example, reconstruction methods based on perturbation theory will have different dependencies on the variations in μ_a and μ_s'. We demonstrate that the characterization is accurate to better than 1% when $D = D_0$ and $\mu_a < 0.1\mu_s'$; when $D = D_{\mu_a}$, characterization errors are approximately 10% for $\mu_a = 0.1\mu_s'$.

2. THEORY

The photon diffusion equation is an approximation of the photon transport equation [Eq. (1)], which is known to describe accurately the propagation of photons through random, turbid, highly scattering media.[1,12,13] We will first consider some general properties of the transport equation. The transport equation is

$$\frac{1}{v}\frac{\partial L(\mathbf{r},\hat{\Omega},t)}{\partial t}+\nabla\cdot L(\mathbf{r},\hat{\Omega},t)\hat{\Omega}+\mu_t L(\mathbf{r},\hat{\Omega},t)=\mu_s\int L(\mathbf{r},\hat{\Omega}',t)f(\hat{\Omega},\hat{\Omega}')\mathrm{d}\hat{\Omega}'+S(\mathbf{r},\hat{\Omega},t), \quad (1)$$

where $L(\mathbf{r},\hat{\Omega},t)$ is the radiance at position r traveling in direction $\hat{\Omega}$ at time t with units of $\mathrm{W\,m^{-2}\,sr^{-1}}$ (sr=steradian=unit solid angle), $f(\hat{\Omega},\hat{\Omega}')$ is the normalized phase function representing the probability of scattering into a solid angle $\hat{\Omega}'$ from angle $\hat{\Omega}$, v is the speed of light in the medium in $\mathrm{cm\,s^{-1}}$, μ_s is the scattering coefficient in $\mathrm{cm^{-1}}$, μ_a is the absorption coefficient in $\mathrm{cm^{-1}}$, $\mu_t=\mu_s+\mu_a$ is the transport coefficient, and $S(\mathbf{r},\hat{\Omega},t)$ is the spatial and angular distribution of the source in $\mathrm{W\,m^{-3}\,sr^{-1}}$.

This equation has an interesting transformation property for the case of homogeneous turbid media:

$$L(\mathbf{r},\hat{\Omega},t)=\exp(-v\mu_a t)L_0(\mathbf{r},\hat{\Omega},t). \quad (2)$$

Here $L(\mathbf{r},\hat{\Omega},t)$ is the solution of the transport equation in the absorbing media, and $L_0(\mathbf{r},\hat{\Omega},t)$ is the solution in the same media, but with zero absorption. The validity of this transformation is checked by substituting Eq. (2) into Eq. (1) and assuming that the source $S(\mathbf{r},\hat{\Omega},t)$ is zero for $r\neq 0$. After substitution we obtain the following absorption-independent transport equation for $L_0(\mathbf{r},\hat{\Omega},t)$:

$$\frac{1}{v}\frac{\partial L_0(\mathbf{r},\hat{\Omega},t)}{\partial t}+\nabla\cdot L_0(\mathbf{r},\hat{\Omega},t)\hat{\Omega}+\mu_s L_0(\mathbf{r},\hat{\Omega},t)=\mu_s\int L_0(\mathbf{r},\hat{\Omega}',t)f(\hat{\Omega},\hat{\Omega}')\mathrm{d}\hat{\Omega}'. \quad (3)$$

We see that the time-dependent impulse response of a homogeneous medium has two separable parts: one that is determined by the scattering coefficient $[L_0(\mathbf{r},\hat{\Omega},t)]$, and one that is determined by the medium absorption $[\exp(-v\mu_a t)]$. This property has been shown to apply for photon propagation in turbid media and has been further exploited by others.[23–26]

In light of this observation, diffusion theory might be expected to have the same transformation property. This follows because the diffusion theory is derived from transport theory, and thus its solutions are a subset of the solutions of the transport equation [Eq. (1)]. The traditional time-dependent photon diffusion equation is[1,12]

$$\nabla^2 U(\mathbf{r},t)-\frac{v\mu_a}{D_{\mu_a}}U(\mathbf{r},t)-\frac{1}{D_{\mu_a}}\frac{\partial U(\mathbf{r},t)}{\partial t}=-\frac{1}{D_{\mu_a}}S(\mathbf{r},t), \quad (4)$$

where $U(\mathbf{r},t)$ is the photon density at position r and time t, v is the speed of light in the medium, $S(\mathbf{r},t)$ is the source term, and D_{μ_a} is the traditional diffusion coefficient

$$D_{\mu_a}=v/3(\mu_s'+\mu_a), \quad (5)$$

in which μ_a is the absorption coefficient and $\mu_s'=\mu_s(1-g)$ is the reduced scattering coefficient, where g is the scattering anisotropy. In analogy with the solutions of the transport equation [Eq. (2)], we might expect that

$$U(\mathbf{r},t)=U_0(\mathbf{r},t)\exp(-v\mu_a t), \quad (6)$$

$$\mathbf{J}(\mathbf{r},t)=\mathbf{J}_0(\mathbf{r},t)\exp(-v\mu_a t), \quad (7)$$

where $U(\mathbf{r},t)$ $[\mathbf{J}(\mathbf{r},t)]$ is the photon density (photon flux) with absorption and $U_0(\mathbf{r},t)$ $[\mathbf{J}_0(\mathbf{r},t)]$ is the photon density (photon flux) satisfying Eq. (4) without absorption $(\mu_a=0)$. When these transformations [Eqs. (6) and (7)] are inserted into the traditional diffusion equation [Eq. (4)], we obtain the following equation for $U_0(\mathbf{r},t)$:

$$\nabla^2 U_0(\mathbf{r},t)-\frac{1}{D_{\mu_a}}\frac{\partial U_0(\mathbf{r},t)}{\partial t}=0. \quad (8)$$

Notice that $U_0(\mathbf{r},t)$ depends on the sample absorption; the independence of the absorption and scattering parts of the solution is unrealized, i.e., the general transformation property is no longer valid.

Now let us review the derivation of the diffusion equation [Eq. (4)] from the transport equation [Eq. (1)] to gain a better understanding of the approximations that led to the breakdown of this transformation property. We start with the following coupled differential equations:

$$\frac{\partial}{\partial t}U(\mathbf{r},t)+\mu_a v U(\mathbf{r},t)+\nabla\cdot\mathbf{J}(\mathbf{r},t)=S_0(\mathbf{r},t), \quad (9)$$

$$\frac{1}{v}\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)+(\mu_s'+\mu_a)\mathbf{J}(\mathbf{r},t)+\frac{v}{3}\nabla U(\mathbf{r},t)=S_1(\mathbf{r},t). \quad (10)$$

These equations are obtained from the transport equation [Eq. (1)] by expanding the radiance and the source term,[13,27] i.e.,

$$L(\mathbf{r},\hat{\Omega},t)=\frac{v}{4\pi}U(\mathbf{r},t)+\frac{3}{4\pi}\mathbf{J}(\mathbf{r},t)\cdot\hat{\Omega}, \quad (11)$$

$$S(\mathbf{r},\hat{\Omega},t)=\frac{1}{4\pi}S_0(\mathbf{r},t)+\frac{3}{4\pi}\mathbf{S}_1(\mathbf{r},t)\cdot\hat{\Omega}. \quad (12)$$

The expansions are substituted into Eq. (1). Integrating Eq. (1) over $\hat{\Omega}$, we obtain Eq. (9). Multiplying Eq. (1) by $\hat{\Omega}$ and integrating over $\hat{\Omega}$, we obtain Eq. (10). This is equivalent to the so-called P_1 approximation of the transport equation.[12,13,15]

At this point it is straightforward to show that the transformations still hold for Eqs. (9) and (10) for a homogeneous, turbid medium. The substitution of the transformation equations (6) and (7) into Eqs. (9) and (10) yields coupled equations for $U_0(\mathbf{r},t)$ and $\mathbf{J}_0(\mathbf{r},t)$ that are still independent of absorption, i.e.,

$$\frac{\partial}{\partial t}U_0(\mathbf{r},t)+\nabla\cdot\mathbf{J}_0(\mathbf{r},t)=0, \quad (13)$$

$$\frac{1}{v}\frac{\partial}{\partial t}\mathbf{J}_0(\mathbf{r},t)+\mu_s'\mathbf{J}_0(\mathbf{r},t)+\frac{v}{3}\nabla U_0(\mathbf{r},t)=0. \quad (14)$$

The next step in the derivation of the diffusion equation is to decouple Eqs. (9) and (10). This is done by using the following substeps: (1) Solve Eq. (9) for $\nabla \cdot \mathbf{J}(\mathbf{r}, t)$, (2) then take the divergence of Eq. (10), and finally (3) substitute $\nabla \cdot \mathbf{J}(\mathbf{r}, t)$ from Eq. (9) into Eq. (10). These substeps produce the telegrapher's equation for $U(\mathbf{r}, t)$ (Ref. 28):

$$-D_{\mu_a}\nabla^2 U(\mathbf{r}, t) + v\mu_a U(\mathbf{r}, t) + \frac{\partial U(\mathbf{r}, t)}{\partial t} + \frac{3D_{\mu_a}}{v}\left[\mu_a \frac{\partial U(\mathbf{r}, t)}{\partial t} + \frac{1}{v}\frac{\partial^2 U(\mathbf{r}, t)}{\partial t^2}\right] = S_0(\mathbf{r}, t) + \frac{3D_{\mu_a}}{v^2}\frac{\partial S_0}{\partial t} - \frac{3D_{\mu_a}}{v}\nabla \cdot \mathbf{S}_1(\mathbf{r}, t). \quad (15)$$

Inserting the separable form for U [transformation equation (6)] into Eq. (15) and canceling terms, we arrive at an equation for $U_0(\mathbf{r}, t)$ that is still independent of absorption:

$$-\nabla^2 U_0(\mathbf{r}, t) + \frac{3\mu_s'}{v}\frac{\partial U_0(\mathbf{r}, t)}{\partial t} + \frac{3}{v^2}\frac{\partial^2 U(\mathbf{r}, t)}{\partial t^2} = 0. \quad (16)$$

Equation (15) is not yet the photon diffusion equation, and the transformation property is still valid. It is interesting to note that for the static case (i.e., no time dependence), the telegrapher's equation [Eq. (15)] reduces to a time-independent diffusion equation with $D = D_{\mu_a}$; however, we will show in Section 3 that this is inconsistent with Monte Carlo results, even for the time-independent case (see Fig. 6 below).

The final step in the derivation of the diffusion equation requires some assumptions in order to drop the underlined terms in Eq. (15).[4,27] The dipole source term $S_1(\mathbf{r}, t)$ is dropped by assuming an isotropic source; this is justified, since collimated sources can be treated as isotropic sources displaced one transport mean free path into the scattering medium from the collimated source. The condition for dropping the remaining terms is most easily understood in the frequency domain. Here the time dependence of the source and the photon density is $\exp(-iwt)$, and the time derivatives can be replaced by $-iw$. The remaining underlined terms are then negligible when[4,27]

$$3wD \ll v^2 \Rightarrow w/\mu_s' v \ll 1. \quad (17)$$

In essence, the photon scattering frequency must be much larger than the source modulation frequency. Furthermore, the albedo should be close to unity for the diffusion approximation to be valid, that is,

$$\mu_s'/(\mu_s' + \mu_a) \approx 1, \quad (18)$$

or

$$\mu_a/(\mu_s' + \mu_a) \ll 1. \quad (19)$$

Thus we are left with Eq. (4). We can no longer separate the absorption and scattering effects [Eqs. (6) and (7)] by using this time-dependent diffusion equation. The transformation property of the transport equation has now been lost! There is no clear physical reason evident from the derivation alone that reveals why these transformations should be lost.

We note, however, that there is an inconsistency in this traditional derivation. When terms of order $\mu_a/(\mu_s' + \mu_a) \ll 1$ [relation (19)] are dropped, then, for the nearly isotropic case (which is a requirement of P_1 expansion), $\mu_a \ll \mu_s'$ and terms of order μ_a/μ_s' should also be dropped; hence

$$D_0 = v/3\mu_s'. \quad (20)$$

This has been discussed in detail by Furutsu and Yamada[18] and Furutsu.[19] The new diffusion coefficient [Eq. (20)] preserves the transformation (6) for the time-dependent diffusion equation (4).[26] We also note that this new result is in agreement with derivations based on the continuum limit of the lattice random-walk theory by Gandjbakche and Weiss.[29]

This is the status of the current controversy in the literature. Rather than considering further the details of the theoretical derivations of the diffusion coefficient, we present Monte Carlo results that exhibit substantially better agreement with diffusion theory when an absorption-independent diffusion coefficient [$D = D_0$; Eq. (20)] is used. The absorption-independent diffusion coefficient thus makes possible characterization of the optical properties of homogeneous media.

3. MONTE CARLO SIMULATIONS

In our Monte Carlo simulations we placed an isotropic photon source at the origin. Concentric spherical detectors were placed around the origin at intervals of 0.2 cm. For each scattering event, a check is made to see if the photon has crossed a detector. If the photon has crossed a spherical shell in the radially outward direction, then the outward flux for that detector at that time is scored. If the crossing was in the inward direction, then the same is done for the inward flux. Photon flux inward and photon flux outward are thus recorded for each detector as a function of time. Figure 1 shows a simple two-dimensional sketch of the setup.

Analytical equations for the diffusion approximation for the inward and outward components of the flux are[13,14,17]

$$J_- = \frac{v}{4}U(\mathbf{r}, t) + \frac{D}{2}\frac{\partial U(\mathbf{r}, t)}{\partial r},$$

$$J_+ = \frac{v}{4}U(\mathbf{r}, t) - \frac{D}{2}\frac{\partial U(\mathbf{r}, t)}{\partial r}, \quad (21)$$

where J_- and J_+ are the radially inward and outward components of the flux, respectively.

Photons are propagated through the medium by using appropriate optical properties and phase functions.[30–32] The Henyey–Greenstein phase function gives the cosine of the scattering angle as

$$\cos\theta = \frac{1}{2g}\left[1 + g^2 - \left(\frac{1 - g^2}{1 - g + 2g\zeta}\right)^2\right], \quad (22)$$

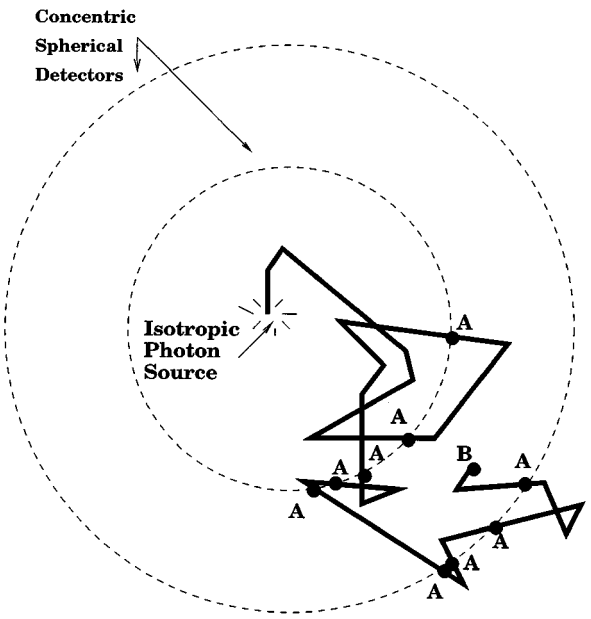

Fig. 1. Simple two-dimensional sketch of the Monte Carlo approach used in counting and propagating photons. Photons are emitted at the origin from an isotropic source. The thick solid curve indicates the path of a single photon that crosses the concentric spherical detectors (indicated by dashed circles) several times. When the photon crosses the detectors (at points **A**), the events are scored. This gives us the inward and outward flux at each detector. The shown photon is absorbed at the point B. Further details are explained in the text, and the Monte Carlo code is available upon request from the authors.

where θ is the scattering angle and ζ is a random number between 0 and 1. The scattering angle in turn gives the direction cosines for the photon. Photons undergo two different interactions: absorption and scattering events. Random numbers are used to give scattering and absorption distances, which are in turn used to decide which event the photon experiences. The photon is then propagated by using the direction cosines and these interaction distances. The details of the process may be found from the Monte Carlo code, which is available on request from the authors, and is similar to previous approaches as documented in the literature.[27,31,32] This process is repeated for each photon until the photon is either absorbed or travels a fixed (large) distance away from the source, which is introduced to limit the simulation time while still being compatible with our assumption of an infinite medium.

The program thus calculates the partial photon flux for each detector and then records the photon density as[14]

$$U(\mathbf{r}, t) = 2(J_- + J_+)/vA\Delta t, \tag{23}$$

where A is the detector surface area $4\pi d^2$ (d is the source–detector separation), v is the speed of light in that medium, and $\Delta t = 20 \times 10^{-12}$ s is the width of the time channel. The photon density is normalized by dividing it by the detector surface area and the width of the time bins. The results are saved in a text file and analyzed by using photon migration imaging software for comparisons with diffusion theory.[33]

4. RESULTS AND DISCUSSION

Photon density at time t and distance r from the source provides an elegant way to examine the absorption dependence of the photon diffusion coefficient D. For an infinite medium, the solution of the photon diffusion equation [Eq. (4)] is[1,34]

$$U(\mathbf{r}, t) = (4\pi Dt)^{-3/2} \exp(-r^2/4Dt - \mu_a v t). \tag{24}$$

If we consider two measurements of $U(\mathbf{r}, t)$ for the same geometry, source–detector position, μ_s', and g, but for different μ_a, then

$$\ln\left[\frac{U_1(\mathbf{r}, t)}{U_2(\mathbf{r}, t)}\right] = \frac{r^2}{4t}\left(\frac{1}{D^2} - \frac{1}{D_1}\right) + (\mu_{a,1} - \mu_{a,2})vt + \ln\left[\left(\frac{D_2}{D_1}\right)^{3/2}\right], \tag{25}$$

where the subscripts 1 and 2 distinguish between the two different media. If $D = D_{\mu_a}$ [Eq. (5)], a plot of Eq. (25) is curved at early time. If $D = D_0$ [Eq. (20)], a plot of Eq. (25) is linear all the time. Figure 2 illustrates this point for the two different diffusion coefficients; the dotted curves indicate the result with $D = D_{\mu_a}$, and the solid line indicates the $D = D_0$ result.

Applying the same analysis to the photon density obtained from the Monte Carlo results, we see that the Monte Carlo simulations indicate clearly that $D = D_0$ as in Eq. (20). Figure 3 shows the Monte Carlo results for $\mu_{a,2} = 0.05$, 0.1, and 0.5 cm^{-1} compared with those for $\mu_{a,1} = 0.005$ cm^{-1}, with a source–detector separation of 1 cm. In all three cases, the plots are straight lines.

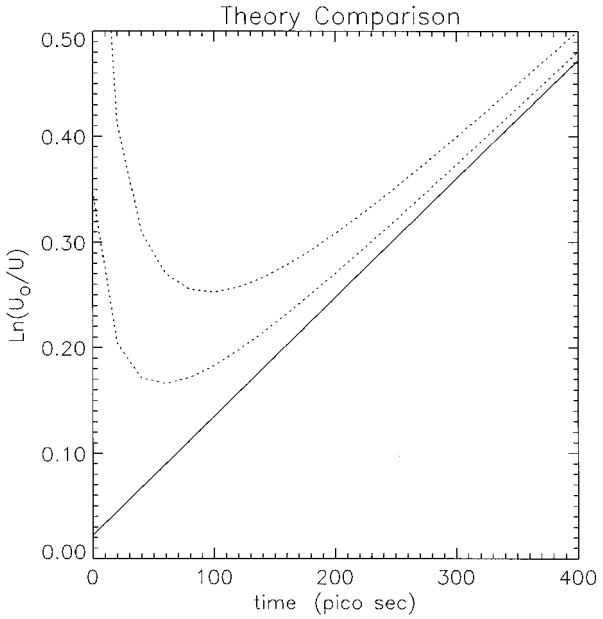

Fig. 2. $\ln[U_1(\mathbf{r}, t)/U_2(\mathbf{r}, t)]$ is plotted for $D = D_0$ (solid line) and for $D = D_{\mu_a}$ (dotted curves). Here $\mu_a = 0$ cm^{-1} for $U_0(\mathbf{r}, t)$, and $\mu_a = 0.05$ cm^{-1} for $U(\mathbf{r}, t)$. Two different source–detector separations, 2 cm and 3 cm, are shown. As explained in the text, the plots for $D = D_{\mu_a}$ are curved and dependent on the source–detector separation, whereas for $D = D_0$ the plots are straight and are not dependent on source–detector separation. $\mu_s' = 10$ cm^{-1} for all cases.

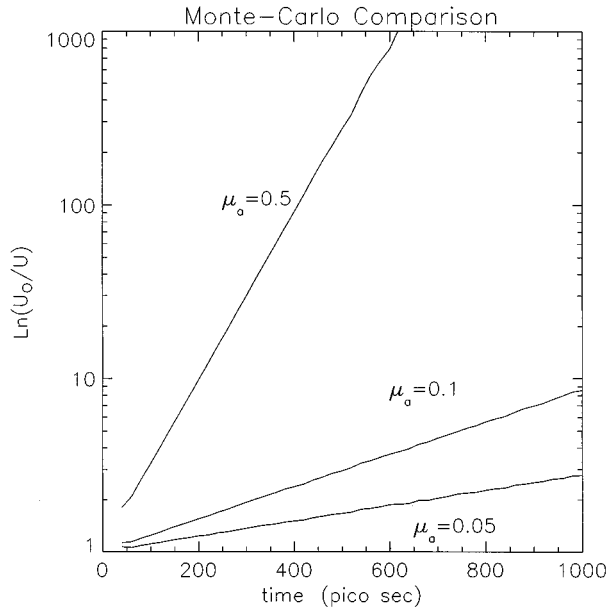

Fig. 3. $\ln[U_0(\mathbf{r}, t)/U(\mathbf{r}, t)]$ is plotted for Monte Carlo results. Here $U_0(\mathbf{r}, t)$ is for $\mu_a = 0.005\ \mathrm{cm}^{-1}$, and $U(\mathbf{r}, t)$ has $\mu_a = 0.05$, 0.1, and 0.5 cm^{-1}, as indicated in the figure. As expected from the theoretical argument for the diffusion coefficient being $D = D_0$, Monte Carlo results produced the expected lines. This clearly indicates that Monte Carlo simulations are indicative of $D = D_0$. $\mu_s' = 10\ \mathrm{cm}^{-1}$, and source–detector separation is 1 cm for all cases shown.

Lines have also been observed for other source–detector separations and scattering and absorption coefficients.

We next investigate the quantitative agreement between the Monte Carlo results and photon diffusion theory by using both $D = D_{\mu_a}$ and $D = D_0$ in the theory. Fourier-transforming the time domain data into the frequency domain, we compare Monte Carlo results with diffusion theory for various modulation frequencies and source–detector separations. We plot the photon density obtained from Monte Carlo simulations and theory results and compare them by looking at the fractional residuals in amplitude, i.e., Amp(Monte Carlo)/Amp(theory). The analysis of the phase residuals does not convey significant quantitative information. In simulating the time evolution of the photons in our Monte Carlo code, we used time bins that were 20 ps wide, i.e., a photon that crossed a detector at 30 ps is recorded to have crossed the detector at the same time as another one that has crossed it at 35 ps. Each time bin contains a temporal average of detected photons, which perturbs the phase in the frequency domain by an unknown amount that increases with increasing frequency. To avoid this inconsistency, we matched the timings by using a free parameter t_0 in the fitting algorithm. This fitting could be avoided in the future by running simulations with smaller time bins, which would require a higher number of photons to get a good signal, or else by quantifying the averaging effect that is due to the temporal bins in a way similar to that of Wang *et al.*[32] Here t_0 is the time at which the laser pulse was introduced into the turbid medium. If the time bins were infinitely small, then t_0 would equal 0. Roughly speaking, t_0 will be one half of the time bin width.

The plots in Fig. 4 show the modulation frequency dependence of the fractional amplitude residuals for an infinite, homogeneous medium with $\mu_s' = 10\ \mathrm{cm}^{-1}$ and source–detector separation of 1 cm for $\mu_a = 0.05$, 0.1, 0.5, and 1.0 cm^{-1}. Figure 4(a) plots the residuals for diffusion theory with the use of $D = D_{\mu_a}$. We see that the agreement between diffusion theory and Monte Carlo results becomes worse with increasing absorption coefficient. Comparing Fig. 4(a) with Fig. 4(b), where $D = D_0$, we see that the residuals are substantially smaller for the latter case.

From Fig. 4 we see that when $D = D_{\mu_a}$, the diffusion theory underestimates the amplitude, whereas, when $D = D_0$, diffusion theory overestimates the amplitude. These differences between theory and experiment result from the diffusion approximation. However, the observed trend suggests that the diffusion approximation can be modified by choosing a diffusion coefficient (D) that has a μ_a dependence but with different weights for the absorption and scattering coefficients. The residuals are flat for all frequencies; therefore they can be corrected by weighting the absorption coefficient, i.e., $D = v/3(\mu_s' + k\mu_a)$ (where k is a scalar coefficient). Figure 4(a) shows that when $D = D_{\mu_a}$, the diffusion approximation is underestimating the amplitude, and from Fig. 4(b) we see that the diffusion approximation overestimates the amplitude when $D = D_0$. This suggests that with $k < 1$ the diffusion approximation can be corrected to match the Monte Carlo results better. However, as we see in Fig. 5, the dependence of the residuals on source–detector separation indicates that a distance-dependent correction is required, and this correction is smaller at large separations when $k = 0$. Such a correction is also inconsistent with the notion that D should be a property of the medium; i.e., it should not be a function of the source–detector separation.

The plots in Fig. 5 show the dependence of the fractional amplitude residuals on the source–detector separation for an infinite, homogeneous medium with $\mu_s' = 10\ \mathrm{cm}^{-1}$ and modulation frequency of 200 MHz for $\mu_a = 0.05$, 0.5, and 1.0 cm^{-1}. Solid curves indicate $D = D_0$, and dotted curves indicate $D = D_{\mu_a}$. It can be seen from these plots that the agreement between diffusion theory and Monte Carlo simulations gets worse with increasing absorption coefficient, as expected. When $D = D_{\mu_a}$, amplitude residuals increase quickly with larger source–detector separations when $\mu_a/\mu_s' > 1/20$. For example, looking at the case with $\mu_a = 1\ \mathrm{cm}^{-1}$, we can see an error that jumps to more than 20% after 1 cm when $D = D_{\mu_a}$, in comparison with errors that are always less than 10% for $D = D_0$. These plots also indicate the breakdown of the diffusion theory near the source. The residuals are not flat anymore, indicating that the photon diffusion coefficient is not a function of absorption, and the correction discussed above is not allowed.

We now consider the effect that the form of the photon diffusion coefficient will have on our ability to characterize accurately the optical properties of turbid media. To this end we have used a least-squares-fitting algorithm to fit for the optical properties obtained by two different methods. We first fit data from a given source–detector

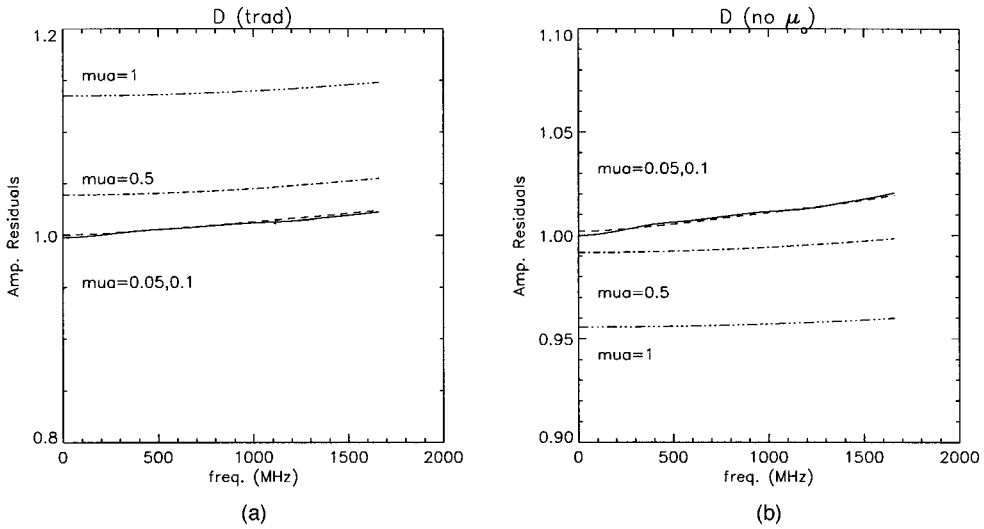

Fig. 4. Fractional amplitude residuals [Amp(Monte Carlo)/Amp(Theory)] versus modulation frequency is shown for different infinite, homogeneous media: (a) results for $D = D_{\mu_a}$, (b) results for $D = D_0$. $\mu_s' = 10\ \mathrm{cm}^{-1}$, and source–detector separation is 1 cm for all cases shown. Media with $\mu_a = 0.05$, 0.1, 0.5, and 1.0 cm^{-1} are shown as indicated in the figure. Comparison of (a) and (b) shows that the agreement is much better when $D = D_0$, which also allows for the possibility of weighting the absorption coefficient, as discussed in the text.

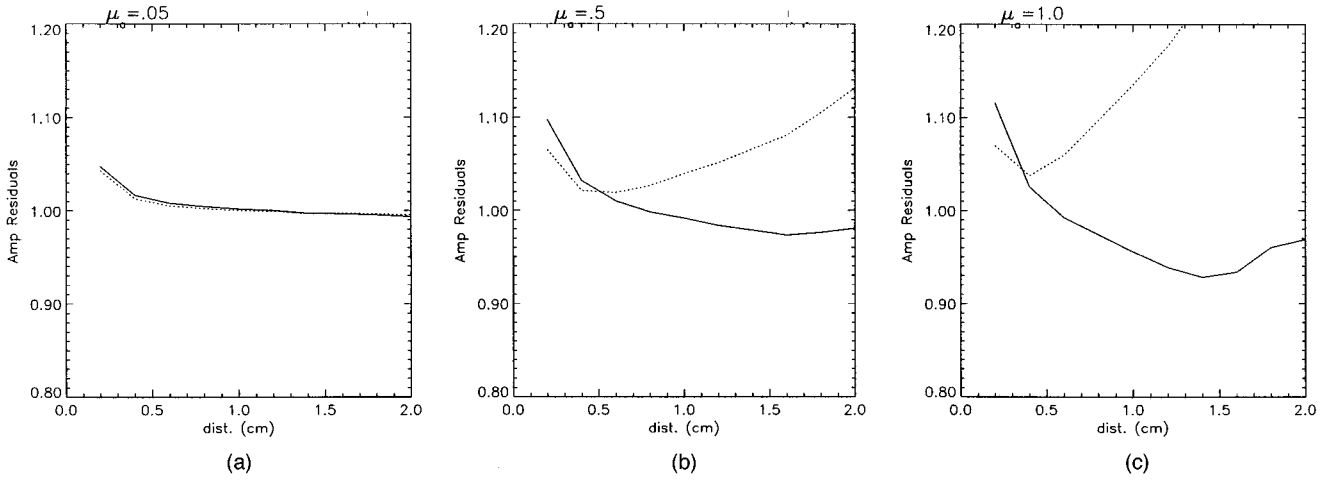

Fig. 5. Fractional amplitude residuals [Amp(Monte Carlo)/Amp(Theory)] versus source–detector separation is shown for different infinite homogeneous media. Solid (dotted) curves indicate results for $D = D_0$ ($D = D_{\mu_a}$). $\mu_s' = 10\ \mathrm{cm}^{-1}$, and modulation frequency = 200 MHz for all cases. $\mu_a = 0.05$, 0.5, and 1.0 cm^{-1} in (a), (b), and (c), respectively. Fractional amplitude residuals are expected to be approximately 1.0 for good agreement. Comparison of the dotted and solid curves show that $D = D_0$ provides much better agreement, as discussed in the text.

separation as a function of modulation frequency. We next fit data for a given modulation frequency over a range of source–detector separations, which is shown in Fig. 6.

When fitting data from a range of modulation frequencies at a given source–detector separation, we see that the reduced scattering coefficient is characterized more accurately when $D = D_0$. However, the absorption coefficient is determined with approximately the same accuracy with either D_0 or D_{μ_a}. This behavior is expected. When fitting data that cover a range of modulation frequencies, one is essentially performing an analysis in the time domain. In the time domain,[1] the absorption coefficient is determined at long times, where the decay of the photon density is dominated by $\exp(-v\mu_a t)$. The determination of μ_a is thus independent of D. Furthermore, the determination of the reduced scattering coefficient using the two different D's differs by the value of μ_a, and $D = D_0$ is more accurate. Here the fitting is accomplished by minimizing the χ^2 value over the range of modulation frequencies for each source–detector separation. We fit simultaneously for the initial amplitude of the source $S_0(\mathbf{r}, t)$ in Eq. (4), to cancel the systematic errors that are observed in Figs. 4 and 5, and for the initial time for the injection of the pulse of light, to cancel systematic errors that are due to temporal binning as explained previously. Both μ_s' and μ_a are also simultaneously fitted for.

When we fit data from a range of source–detector separations for a given frequency (Fig. 6), we see that the

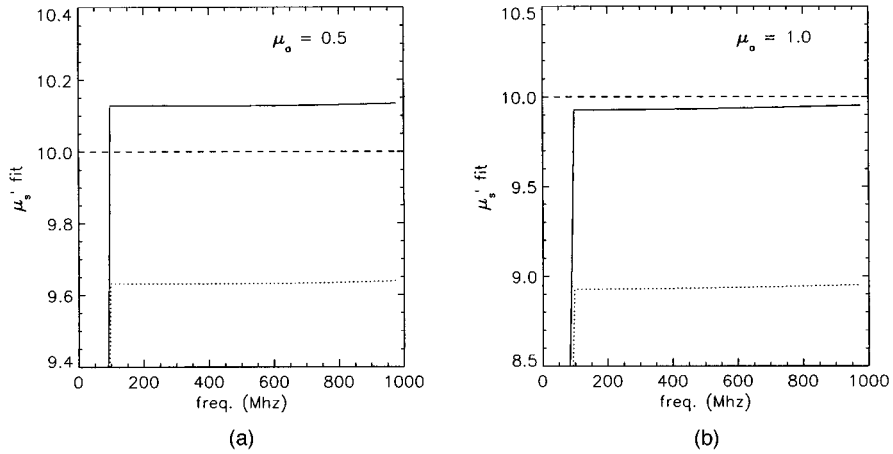

Fig. 6. μ_s fitting results through a range of source–detector separations (1 cm to 2 cm) versus modulation frequency is shown for different absorption coefficients. Both μ_s' and μ_a are fitted simultaneously at each modulation frequency, except for $w = 0$ (i.e., dc case). In the latter case, μ_a is assumed to be known, and we fit for μ_s'. Solid (dotted) lines indicate results for $D = D_0$ ($D = D_{\mu_a}$). $\mu_s' = 10\ \mathrm{cm}^{-1}$; $\mu_a = 0.05$ and $1.0\ \mathrm{cm}^{-1}$ in (a) and (b), respectively. Dashed lines are used to indicate the expected μ_s. We see that for $\mu_a = 0.5$ and $1.0\ \mathrm{cm}^{-1}$ the accuracy for $D = D_0$ is within 2%, whereas when $D = D_{\mu_a}$, the discrepancy is as large as more than 10%. We also note that, even for the time-independent case, $D = D_0$ gives better accuracy. For low μ_a values investigated, such as $\mu_a = 0.05$ and $0.1\ \mathrm{cm}^{-1}$, we obtain results accurate within 1%, which is under the expected noise level. The discrepancies and the comparison of the results for two different D's are not significant. Here, once again, the details of D do not play a role, and hence the plots for the μ_a fits are not shown here. The details are discussed in the text.

breakdown of the diffusion approximation near sources plays a role in the comparison of the fit results from two different D's. We overcome this problem by choosing a fitting range from 1 cm to 2 cm, where the diffusion theory breakdown region is avoided. Both μ_s' and μ_a are fitted simultaneously at each modulation frequency except for $w = 0$ (i.e., dc case). In the latter case, μ_a is assumed to be known, and we fit for μ_s'. For low μ_a, i.e., $\mu_a \leqslant 0.1\ \mathrm{cm}^{-1}$, we obtain fits for μ_s' accurate to within 1% by using both D_0 and D_{μ_a}. In Fig. 6 we see that for $\mu_a \geqslant 0.5\ \mathrm{cm}^{-1}$ the accuracy for $D = D_0$ is better than that for $D = D_{\mu_a}$ (i.e., $<2\%$ versus $>10\%$). Thus we also note that even for the time-independent case we find that $D = D_0$ gives better accuracy. When we fitted for μ_a, the form of D did not change the fitting accuracy, as expected.

The data shown in the figures are results from simulations with anisotropy constant $g = 0$. However, they describe different values of g equally well. In our simulations, anisotropy constants of 0.001, 0.2, 0.5, and 0.9 all produced similar results.

5. SUMMARY

We have confirmed that diffusion theory is accurate for infinite, homogeneous, turbid media in describing the nature of the propagation of photons. However, the traditionally accepted photon diffusion coefficient is shown to be a poor approximation by comparison with (the analytically derived) $D_0 = v/3\mu_s'$. The discrepancy between Monte Carlo results and diffusion theory is found to be within 5% for absorption-independent $D = D_0$, whereas it is as high as 15%–20% when $D = D_{\mu_a}$ is absorption dependent for $\mu_a \leqslant 0.1\mu_s'$. This is important for optical spectroscopy, where the accurate description of photon propagation in turbid media is necessary for obtaining quantitative results. A comparison of the accuracy of fitting for the optical properties of infinite, homogeneous, turbid media is made between two forms of the diffusion coefficient. Good accuracy of within 10% is observed in both cases, with absorption-independent $D = D_0$ giving better agreement (within 2%), whereas absorption-dependent $D = D_{\mu_a}$ is accurate only within 10% for $\mu_a \leqslant 0.1\mu_s'$.

Note added in proof: After submission of this paper we learned of other papers focusing on related issues and arriving at similar conclusions. These papers are by Bassani *et al.*[35] and Nishimura *et al.*[36]

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