

## Chapter 5

### Imaging the Scattering coefficient

In the human body there are localized absorption changes and changes in the reduced scattering coefficient. Scattering is caused by a mismatch in the index of refraction and depends on the size and shape of the scattering object. The reduced scattering coefficient also depends on the density of scatterers. The presence of certain solutes such as glucose or potassium can change the index of refraction of the intra- or extra-cellular fluid, changing the reduced scattering coefficient. Imaging the scattering coefficient is important not only because large scattering changes can easily corrupt an absorption image, but also because scattering is an additional form of contrast. We will use the same expansions (Born, Rytov) that we used for the absorption case and the same inversion techniques (SVD, SIRT) to image the reduced scattering coefficient.

To solve the diffusion equation for an infinite, heterogeneously scattering medium we expand the diffusion coefficient,  $D = 1/3\mu'_s$ , into spatially dependent,  $\delta D(\mathbf{r})$ , and independent (background),  $D_o$ , pieces, i.e.

$$D(\mathbf{r}) \Rightarrow \delta D(\mathbf{r}) + D_o. \quad (5.1)$$

These terms are then incorporated into the diffusion equation (equation 2.18),

$$\mathbf{J}_{ac}(\mathbf{r}) \approx -D\nabla U_{ac}(\mathbf{r}) \quad (5.2)$$

to obtain

$$-i\omega U(\mathbf{r}, \mathbf{r}_s) + v\mu_a U(\mathbf{r}, \mathbf{r}_s) + \nabla \cdot \mathbf{J}(\mathbf{r}, \mathbf{r}_s) = B\delta(\mathbf{r}_s), \quad (5.3)$$

$$\mathbf{J}(\mathbf{r}, \mathbf{r}_s) = -(\delta D(\mathbf{r}) + D_o)\nabla U(\mathbf{r}, \mathbf{r}_s). \quad (5.4)$$

When we insert equation 5.4 into 5.3, we obtain

$$-i\omega U(\mathbf{r}, \mathbf{r}_s) + v\mu_a U(\mathbf{r}, \mathbf{r}_s) - \nabla \cdot ((\delta D(\mathbf{r}) + D_o)\nabla U(\mathbf{r}, \mathbf{r}_s)) = B\delta(\mathbf{r}_s). \quad (5.5)$$

Then divide through by  $-D_o$  we have

$$(\nabla^2 + k^2)U(\mathbf{r}, \mathbf{r}_s) + \nabla F(\mathbf{r}) \cdot \nabla U(\mathbf{r}, \mathbf{r}_s) + F(\mathbf{r})\nabla^2 U(\mathbf{r}, \mathbf{r}_s) = -B\delta(\mathbf{r}_s)/D_o \quad (5.6)$$

$$F(\mathbf{r}) = \delta D(\mathbf{r})/D_o. \quad (5.7)$$

### 5.1 Born Expansion

To solve this heterogeneous diffusion equation, we will use the same approaches as we described in Chapter 4. We will assume that we have a homogeneous system. A finite system can be handled using image sources, just as we discussed in the absorption case. First, using the Born approximation,

$$U(\mathbf{r}, \mathbf{r}_s) = U_o(\mathbf{r}, \mathbf{r}_s) + U_{sc}(\mathbf{r}, \mathbf{r}_s) \quad (5.8)$$

so

$$\begin{aligned} & \underbrace{(\nabla^2 + k^2)U_o(\mathbf{r}, \mathbf{r}_s)}_{\text{homogeneous}} + (\nabla^2 + k^2)U_{sc}(\mathbf{r}, \mathbf{r}_s) \quad (5.9) \\ & + \nabla F(\mathbf{r}) \cdot \nabla (U_o(\mathbf{r}, \mathbf{r}_s) + U_{sc}(\mathbf{r}, \mathbf{r}_s)) + F(\mathbf{r})\nabla^2 (U_o(\mathbf{r}, \mathbf{r}_s) + U_{sc}(\mathbf{r}, \mathbf{r}_s)) \\ & = \underbrace{-B\delta(\mathbf{r}_s)/D_o}_{\text{homogeneous}}. \quad (5.10) \end{aligned}$$

If we subtract off the homogeneous diffusion equation (marked *homogeneous*) and assume that  $U_{sc} \ll U_o$ , we again arrive at a Helmholtz equation for  $U_{sc}$ ,

$$(\nabla^2 + k^2)U_{sc}(\mathbf{r}, \mathbf{r}_s) = -\nabla F(\mathbf{r}) \cdot \nabla U_o(\mathbf{r}, \mathbf{r}_s) - F(\mathbf{r})\nabla^2 U_o(\mathbf{r}, \mathbf{r}_s) \quad (5.11)$$

which can be solved using the Green function method,

$$U_{sc}(\mathbf{r}_d, \mathbf{r}_s) = - \int d^3r G(\mathbf{r} - \mathbf{r}_d) \left( \nabla F(\mathbf{r}) \cdot \nabla U_o(\mathbf{r}, \mathbf{r}_s) + F(\mathbf{r})\nabla^2 (U_o(\mathbf{r}, \mathbf{r}_s)) \right) \quad (5.12)$$

$$G(\mathbf{r} - \mathbf{r}_d) = \frac{\exp(-|\mathbf{r} - \mathbf{r}_d|)}{4\pi|\mathbf{r} - \mathbf{r}_d|}. \quad (5.13)$$

Using Green's first identity [65],

$$\int_V d^3x \phi \nabla^2 \psi = \int_S d^2x \phi \mathbf{n} \cdot \nabla \psi - \int_V d^3x \nabla \phi \cdot \nabla \psi, \quad (5.14)$$

equation 5.12 becomes,

$$\begin{aligned} U_{sc}(\mathbf{r}_d, \mathbf{r}_s) &= - \underbrace{\int_V d\mathbf{r} G(\mathbf{r} - \mathbf{r}_d) \nabla F(\mathbf{r}) \cdot \nabla U_o(\mathbf{r}, \mathbf{r}_s)}_I \\ &\quad - \underbrace{\int_S da U_o(\mathbf{r}, \mathbf{r}_s) \mathbf{n} \cdot \nabla (G(\mathbf{r} - \mathbf{r}_d) F(\mathbf{r}))}_II \\ &\quad + \underbrace{\int_V d\mathbf{r} \nabla U_o(\mathbf{r}, \mathbf{r}_s) \cdot \nabla (G(\mathbf{r} - \mathbf{r}_d) F(\mathbf{r}))}_III. \end{aligned} \quad (5.15)$$

The integral marked II is equal to zero, since we can take this surface to be at infinity, and both  $U_o$  and  $G$  exponentially decay to zero at infinity. The third integral can be expanded,

$$\begin{aligned} \int_V d\mathbf{r} \nabla U_o(\mathbf{r}, \mathbf{r}_s) \cdot \nabla (G(\mathbf{r} - \mathbf{r}_d) F(\mathbf{r})) &= \int_V d\mathbf{r} \nabla U_o(\mathbf{r}, \mathbf{r}_s) \cdot \nabla G(\mathbf{r} - \mathbf{r}_d) F(\mathbf{r}) \\ &\quad + \int_V d\mathbf{r} \nabla U_o(\mathbf{r}, \mathbf{r}_s) \cdot \nabla F(\mathbf{r}) G(\mathbf{r} - \mathbf{r}_d). \end{aligned} \quad (5.16)$$

Note that the second term in the expansion cancels with the term marked I in equation 5.15 and finally

$$U_{sc}(\mathbf{r}_d, \mathbf{r}_s) = \int_V d\mathbf{r} \nabla U_o(\mathbf{r}, \mathbf{r}_s) \cdot \nabla G(\mathbf{r} - \mathbf{r}_d) \frac{\delta D(\mathbf{r})}{D_o}. \quad (5.17)$$

## 5.2 Rytov Expansion

Just as we did in the absorption case, we can also make a Rytov approximation,

$$U(\mathbf{r}, \mathbf{r}_s) = U_o(\mathbf{r}, \mathbf{r}_s) U_{sc}(\mathbf{r}, \mathbf{r}_s) = \exp(\phi_o(\mathbf{r}, \mathbf{r}_s) + \phi_{sc}(\mathbf{r}, \mathbf{r}_s)), \quad (5.18)$$

$$U_o(\mathbf{r}, \mathbf{r}_s) = \exp(\phi_o(\mathbf{r}, \mathbf{r}_s)). \quad (5.19)$$

For notational ease we will define,

$$U_o \equiv U_o(\mathbf{r}, \mathbf{r}_s); \phi_o \equiv \phi_o(\mathbf{r}, \mathbf{r}_s) \quad (5.20)$$

$$U_{sc} \equiv U_{sc}(\mathbf{r}, \mathbf{r}_s); \phi_{sc} \equiv \phi_{sc}(\mathbf{r}, \mathbf{r}_s). \quad (5.21)$$

When we plug these terms into the diffusion equation (5.6) we arrive at,

$$(\nabla^2 + k^2)(U_o U_{sc}) + \nabla F \cdot \nabla(U_o U_{sc}) + F \nabla^2(U_o U_{sc}) = -B\delta(\mathbf{r}_s)/D_o \quad (5.22)$$

$$F = \delta D(\mathbf{r})/D_o. \quad (5.23)$$

We can expand the gradients of products,

$$\begin{aligned} & \nabla^2 U_o U_{sc} + \nabla^2 U_{sc} U_o + 2\nabla U_o \cdot \nabla U_{sc} + k^2 U_o U_{sc} + \nabla F \cdot \nabla U_o U_{sc} \quad (5.24) \\ & + \nabla F \cdot \nabla U_{sc} U_o + F \nabla^2 U_o U_{sc} + F \nabla^2 U_{sc} U_o + 2F \nabla U_o \cdot \nabla U_{sc} = -B\delta(\mathbf{r}_s)/D_o \end{aligned}$$

Note that

$$\nabla U_{sc} = \nabla \phi_{sc} U_{sc}. \quad (5.25)$$

Using these identities and dividing through by  $U_{sc}$ , Equation 5.24 becomes

$$\begin{aligned} & \underbrace{\nabla^2 U_o}_{\text{homogeneous}} + \nabla^2 \phi_{sc} U_o + (\nabla \phi_{sc})^2 U_o + 2\nabla U_o \cdot \nabla \phi_{sc} + \underbrace{k^2 U_o}_{\text{homogeneous}} \quad (5.26) \\ & + \nabla F \cdot \nabla U_o + \nabla F \cdot \nabla \phi_{sc} U_o + F \nabla^2 U_o + F \nabla^2 \phi_{sc} U_o \\ & + F (\nabla \phi_{sc})^2 U_o + 2F \nabla U_o \cdot \nabla \phi_{sc} = \underbrace{-B\delta(\mathbf{r}_s)/D_o}_{\text{homogeneous}} \end{aligned}$$

As in the absorption case we have assumed that  $U_{sc}$  is negligible at the source position. Next we subtract the homogeneous diffusion equation (marked *homogeneous*) and note that

$$\nabla^2(U_o \phi_{sc}) = \nabla^2 U_o \phi_{sc} + \nabla^2 \phi_{sc} U_o + 2\nabla \phi_{sc} \cdot \nabla U_o \quad (5.27)$$

$$= -k^2 U_o \phi_{sc} + \nabla^2 \phi_{sc} U_o + 2\nabla \phi_{sc} \cdot \nabla U_o \quad (5.28)$$

We use this equation to replace  $2\nabla \phi_{sc} \cdot \nabla U_o$  in equation 5.26

$$\underbrace{\nabla^2 \phi_{sc} U_o}_{+\alpha} + (\nabla \phi_{sc})^2 U_o + \underbrace{\nabla^2(U_o \phi_{sc}) + k^2 U_o \phi_{sc}}_{\text{Helmholtz-like}} \quad (5.29)$$

$$\underbrace{-\nabla^2 \phi_{sc} U_o}_{-\alpha} + \nabla F \cdot \nabla U_o + \nabla F \cdot \nabla \phi_{sc} U_o + F \nabla^2 U_o \quad (5.30)$$

$$+ F \nabla^2 \phi_{sc} U_o + F (\nabla \phi_{sc})^2 U_o + 2F \nabla U_o \cdot \nabla \phi_{sc} = 0. \quad (5.31)$$

This equation can be rearranged to form another Helmholtz equation. This Helmholtz equation is solved by convolving with the appropriate Green function,

$$U_o(\mathbf{r}_d, \mathbf{r}_s)\phi_{sc}(\mathbf{r}_d, \mathbf{r}_s) = - \int_V d^3r G \left\{ (\nabla\phi_{sc})^2 U_o + \underbrace{\nabla F \cdot \nabla U_o}_A + \underbrace{F \nabla^2 U_o}_I \right. \quad (5.32)$$

$$\left. + \underbrace{\nabla F \cdot \nabla \phi_{sc} U_o}_B + \underbrace{F \nabla^2 \phi_{sc} U_o}_{II} + F(\nabla\phi_{sc})^2 U_o + \underbrace{2F \nabla U_o \cdot \nabla \phi_{sc}}_C \right\},$$

where

$$G \equiv G(\mathbf{r}_d - r) \quad (5.33)$$

If we again use Green's first identity 5.14 then the term marked I becomes

$$\int_V d^3x FG \nabla^2 U_o = \int_S d^2x \underbrace{(FG)\mathbf{n} \cdot \nabla U_o}_{=0} - \int_V d^3x \nabla(FG) \cdot \nabla U_o \quad (5.34)$$

$$= - \int_V d^3x \underbrace{G \nabla F \cdot \nabla U_o}_A - \int_V d^3x F \nabla G \cdot \nabla U_o, \quad (5.35)$$

and the resulting term marked A cancels with the term marked A in equation 5.32.

Similarly, the term marked II becomes

$$\int_V d^3x GFU_o \nabla^2 \phi_{sc} = \int_S d^2x \underbrace{GFU_o \mathbf{n} \cdot \nabla \phi_{sc}}_{=0} - \int_V d^3x \nabla(GFU_o) \cdot \nabla \phi_{sc} \quad (5.36)$$

$$= - \int_V d^3x \underbrace{GF \nabla U_o \cdot \nabla \phi_{sc}}_C - \int_V d^3x \underbrace{GU_o \nabla F \cdot \nabla \phi_{sc}}_B - \int_V d^3x FU_o \nabla G \cdot \nabla \phi_{sc} \quad (5.37)$$

and the resulting term marked B cancels with the term marked B in equation 5.32.

Likewise with C. This leaves us with

$$U_o(\mathbf{r}_d, \mathbf{r}_s)\phi_{sc}(\mathbf{r}_d, \mathbf{r}_s) = - \int_V d\mathbf{r} G(\nabla\phi_{sc})^2 U_o - F \nabla G \cdot \nabla U_o - FU_o \nabla G \cdot \nabla \phi_{sc} \quad (5.38)$$

$$+ FG(\nabla\phi_{sc})^2 U_o + FG \nabla U_o \cdot \nabla \phi_{sc}. \quad (5.39)$$

Note that because our system is totally symmetric with respect to the source and detector,

$$\int d\mathbf{r} FU_o(\mathbf{r}, \mathbf{r}_s) \nabla G(\mathbf{r}_d - \mathbf{r}) \cdot \nabla \phi_{sc} = \int d\mathbf{r} FU_o(\mathbf{r}, \mathbf{r}_s) \nabla G(\mathbf{r}_d - \mathbf{r}) \cdot \nabla \phi_{sc} \quad (5.40)$$

$$= \int d\mathbf{r} F \frac{\exp(ik|\mathbf{r} - \mathbf{r}_s|)}{4\pi D_o |\mathbf{r} - \mathbf{r}_s|} \nabla \frac{\exp(ik|\mathbf{r}_d - \mathbf{r}|)}{4\pi |\mathbf{r}_d - \mathbf{r}|} \cdot \nabla \phi_{sc} \quad (5.41)$$

$$= \int d\mathbf{r} F \frac{\exp(ik|\mathbf{r}_d - \mathbf{r}|)}{4\pi D_o |\mathbf{r}_d - \mathbf{r}|} \nabla \frac{\exp(ik|\mathbf{r} - \mathbf{r}_s|)}{4\pi |\mathbf{r} - \mathbf{r}_s|} \cdot \nabla \phi_{sc} \quad (5.42)$$

$$= \int d\mathbf{r} FG(r_d - \mathbf{r}) \nabla U_o(\mathbf{r}, \mathbf{r}_s) \cdot \nabla \phi_{sc}. \quad (5.43)$$

If we assume that

$$(\nabla \phi_{sc})^2 \ll \nabla G \cdot \nabla U_o \quad (5.44)$$

and  $F$  less than or on the order of 1, we can now write equation 5.32 as

$$U_o(\mathbf{r}_d, \mathbf{r}_s) \phi_{sc}(\mathbf{r}_d, \mathbf{r}_s) = \int_V d\mathbf{r} F \nabla G \cdot \nabla U_o. \quad (5.45)$$

Note that in most systems of particular interest in the body, the reduced scattering coefficient is not expected to vary more than 100%. So our assumption that  $F$  is less than or on the order of 1 is a good assumption.

$$F = \delta D / D \quad (5.46)$$

$$= (D - D_o) / D \quad (5.47)$$

$$= 1 - (\mu'_s + \delta \mu'_s) / \mu'_s \quad (5.48)$$

$$= \delta \mu'_s / \mu'_s \quad (5.49)$$

As in the absorption case, the structure of the Born and Rytov solutions look very similar. Again, we see that the Born approximation makes the assumption that the scattered wave is small, and the scattered wave scales linearly with the absorption. The Rytov approximation does not place a restriction on the magnitude of the scattered wave change, but rather assumes that the scattered field is slowly varying. In the absorption case, a calibration plot of reconstructed absorption versus true absorption showed that the Rytov solution procured an better reconstruction of the absorption. In the scattering case, the Rytov solution does not improve the calibration plot. Virmont and Ledanois [66] have studied this effect, and have suggested an adjustment to the Rytov solution that improves the scattering images.

### 5.3 Matrix Equations

We now have an expression which relates the scattered wave to the heterogeneous optical properties. From here the analysis is the same as in the absorption case; we will digitize the integral, and make a series of measurements to generate a matrix.

Born:

$$\begin{pmatrix} U_{sc}(\mathbf{r}_{s1}, \mathbf{r}_{d1}) \\ \vdots \\ U_{sc}(\mathbf{r}_{sm}, \mathbf{r}_{dm}) \end{pmatrix} = \begin{pmatrix} W_{11}^B & \dots & W_{1n}^B \\ \vdots & \ddots & \vdots \\ W_{m1}^B & \dots & W_{mn}^B \end{pmatrix} \begin{pmatrix} \delta D(\mathbf{r}_1) \\ \vdots \\ \delta D(\mathbf{r}_n) \end{pmatrix}$$

$$W_{ij}^B = \nabla U_o(\mathbf{r}_j, \mathbf{r}_{si}) \cdot \nabla G(\mathbf{r}_{di} - \mathbf{r}_j) v h^3 / D_o \quad (5.50)$$

Rytov:

$$\begin{pmatrix} \phi_{sc}(\mathbf{r}_{s1}, \mathbf{r}_{d1}) \\ \vdots \\ \phi_{sc}(\mathbf{r}_{sm}, \mathbf{r}_{dm}) \end{pmatrix} = \begin{pmatrix} W_{11}^R & \dots & W_{1n}^R \\ \vdots & \ddots & \vdots \\ W_{m1}^R & \dots & W_{mn}^R \end{pmatrix} \begin{pmatrix} \delta D(\mathbf{r}_1) \\ \vdots \\ \delta D(\mathbf{r}_n) \end{pmatrix}$$

$$W_{ij}^R = \frac{\nabla U_o(\mathbf{r}_j, \mathbf{r}_{si}) \cdot \nabla G(\mathbf{r}_{di} - \mathbf{r}_j) v h^3}{U_o(\mathbf{r}_{di}, \mathbf{r}_{si}) D_o} \quad (5.51)$$

The matrix is inverted using the same algorithms; SVD or SIRT to obtain  $\delta D$

Figure 5.1 demonstrates the reconstruction of a single spherical object (1.2 cm in diameter) from *experimental data*. The background media has  $\mu_a^o = 0.023 \text{ cm}^{-1}$  and  $\mu_s^o = 6.0 \text{ cm}^{-1}$ . A single, resin sphere having the same absorption coefficient as the surrounding medium, but a higher scattering coefficient ( $\mu_s' \approx 15.0 \text{ cm}^{-1}$ ), is imaged using 120 measurements of amplitude and phase. In this reconstruction, we have made use of *a priori* knowledge that the object is either absorbing or scattering, that is we have assumed that the absorption coefficient is homogeneous throughout

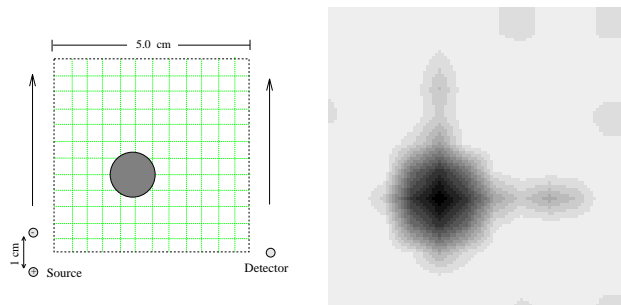


Figure 5.1: The reconstruction of a highly scattering sphere using 1000 SIRT iterations. See text for further discussion.

the medium. In chapter 6 we discuss solving for both absorption and scattering simultaneously.

Just as in the absorption case, we have found that as we continue to iterate, the image of a sphere gets gradually smaller and more highly scattering. Boas *et al.* [22] have demonstrated that even in a best case scenario, the difference between a small, highly scattering object and a larger, less scattering object is practically immeasurable for objects with a diameter of less than 1 cm [22]. The authors demonstrate that there is a family of degenerate solutions which all conserve the quantity  $\delta\mu_a v$  where  $v$  is the volume of the sphere. Thus, in the SIRT reconstructions, the consecutive iterations move the solution through this family of solutions. Because we always start from the same initial guess (a homogeneous system) the reconstruction always moves through the family of solutions in the same way. Figure 5.2 demonstrates a series of reconstructions for different numbers of iterations. Note that as the number of iterations increases, the reduces scattering coefficient of the object increases, and the size decreases. We have left the iteration number as a free parameter in our reconstructions.

If we use a finite media, we must adjust the weights in our calculation to reflect the new boundary conditions. In particular, we would use the same methodology as we did in section 4.13; replace  $U_o$  and  $G$  using the appropriate Green function solutions for the given boundary condition.



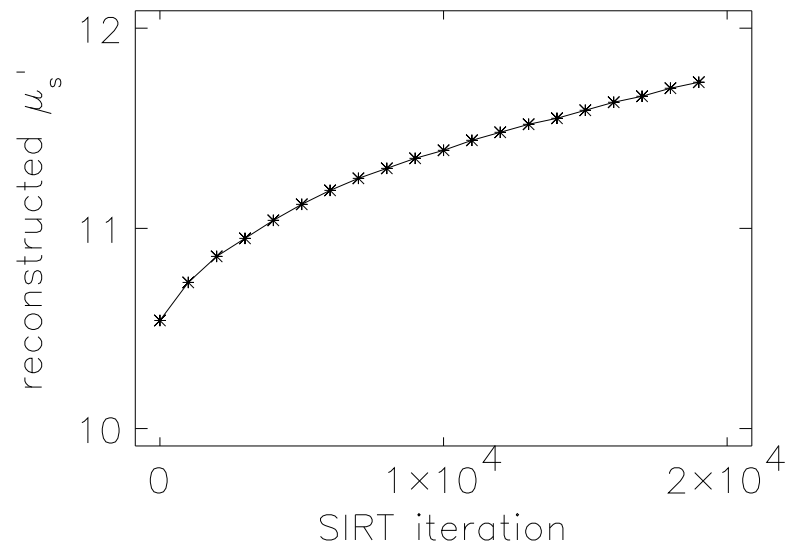


Figure 5.2: The reconstructed reduced scattering coefficient (maximum value) as a function of iteration. A 1 cm diameter sphere with  $\mu'_s = 12 \text{ cm}^{-1}$ ,  $\mu_a = 0.03 \text{ cm}^{-1}$  is embedded with a medium with  $\mu'_s = 10 \text{ cm}^{-1}$ ,  $\mu_a = 0.03 \text{ cm}^{-1}$ . The sources scan the sides of a 7 cm square with a source modulation frequency of 200 MHz.

